

# RIEMANN HYPOTHESIS AND THE ARC LENGTH OF THE RIEMANN $Z(t)$ -CURVE

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ABSTRACT. On Riemann hypothesis it is proved in this paper that the arc length of the Riemann  $Z$ -curve is asymptotically equal to the double sum of local maxima of the function  $Z(t)$  on corresponding segment. This paper is English remake of our paper [9], with short appendix concerning new integral generated by Jacob's ladders added.

## 1. INTRODUCTION AND RESULT

1.1. Main object of this paper is the study of the integral

$$(1.1) \quad \int_T^{T+H} \sqrt{1 + \{Z'(t)\}^2} dt,$$

i.e. the study of the arc length of the Riemann curve

$$y = Z(t), \quad t \in [T, T+H], \quad T \rightarrow \infty,$$

where (see [13], pp. 79, 329)

$$(1.2) \quad \begin{aligned} Z(t) &= e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + it\right), \\ \vartheta(t) &= -\frac{t}{2} \ln \pi + \operatorname{Im} \ln \Gamma\left(\frac{1}{4} + i\frac{t}{2}\right) = \\ &= \frac{t}{2\pi} \ln \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \mathcal{O}\left(\frac{1}{t}\right). \end{aligned}$$

*Remark 1.* Let us remind that the formula

$$(1.3) \quad \begin{aligned} \{Z(t)\} &= e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + it\right) = \\ &= 2 \sum_{n \leq \sqrt{t}} \frac{1}{\sqrt{n}} \cos\{\vartheta(t) - t \ln n\} + \mathcal{O}(t^{-1/4}), \quad \bar{t} = \sqrt{\frac{t}{2\pi}} \end{aligned}$$

was known to Riemann (see [11], p. 60, comp. [12], p. 98).

Next, we will denote the roots of the equations

$$Z(t) = 0, \quad Z'(t) = 0, \quad t_0 \neq \gamma$$

by the symbols

$$\{\gamma\}, \quad \{t_0\},$$

correspondingly.

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*Key words and phrases.* Riemann zeta-function.

*Remark 2.* On the Riemann hypothesis, the points of the sequences  $\{\gamma\}$  and  $\{t_0\}$  are separated each from other (see [3], Corollary 3), i.e. in this case we have

$$\gamma' < t_0 < \gamma'',$$

where  $\gamma', \gamma''$  are neighboring points of the sequence  $\{\gamma\}$ . Of course,  $Z(t_0)$  is local extremum of the function  $Z(t)$  located at  $t = t_0$ .

1.2. In this paper we use the Riemann hypothesis together with some synthesis of properties of the sequences

$$\{t_0\}, \{h_\nu(\tau)\},$$

where the numbers  $h_\nu(\tau)$  are defined by the equation (comp. (1.2))

$$\begin{aligned} \vartheta_1[h_\nu(\tau)] &= \pi\nu + \tau + \frac{\pi}{2}, \quad \nu = 1, 2, \dots, \quad \tau \in [-\pi, \pi], \\ (1.4) \quad \vartheta_1(t) &= \frac{t}{2} \ln \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8}, \\ \vartheta(t) &= \vartheta_1(t) + \mathcal{O}\left(\frac{1}{t}\right), \end{aligned}$$

in order to obtain the following theorem.

**Theorem.** On the Riemann hypothesis we have the asymptotic formula

$$\begin{aligned} (1.5) \quad \int_T^{T+H} \sqrt{1 + \{Z'(t)\}^2} dt &= 2 \sum_{T \leq t_0 \leq T+H} |Z(t_0)| + \\ &+ \Theta H + \mathcal{O}\left(T^{\frac{1}{\ln \ln T}}\right), \\ \Theta &= \Theta(T, H) \in (0, 1), \quad H = T^\epsilon, \quad T \rightarrow \infty \end{aligned}$$

for every fixed  $\epsilon > 0$ .

*Remark 3.* Geometric meaning of our asymptotic formula (1.5) is as follows: the arc length of the Riemann curve

$$y = Z(t), \quad t \in [T, T+H]$$

is asymptotically equal to the double of the sum of local maxima of the function

$$|Z(t)|, \quad t \in [T, T+H].$$

## 2. DISCRETE FORMULAE – LEMMA 1

2.1. In this part of the paper we use the following formula

$$\begin{aligned} (2.1) \quad Z'(t) &= -2 \sum_{n < P} \frac{1}{\sqrt{n}} \ln \frac{P}{n} \sin\{\vartheta - t \ln n\} + \\ &+ \mathcal{O}(T^{-1/4} \ln T), \quad P = \sqrt{\frac{T}{2\pi}}, \end{aligned}$$

that we have obtained in our work [6], (see (2.1)). Next, we obtain from (2.1) in the case

$$\vartheta \rightarrow \vartheta_1$$

(see (1.4)) that

$$(2.2) \quad \begin{aligned} Z'(t) = & -2 \sum_{n < P} \frac{1}{\sqrt{n}} \ln \frac{P}{n} \sin\{\vartheta_1 - t \ln n\} + \\ & + \mathcal{O}(T^{-1/4} \ln T), \quad H \in (0, \sqrt[4]{T}]. \end{aligned}$$

Let  $S(a, b)$  denotes elementary trigonometric sum

$$S(a, b) = \sum_{a \leq n \leq b} n^{it}, \quad 1 \leq a < b \leq 2a, \quad b \leq \sqrt{\frac{t}{2\pi}}.$$

Then we obtain from (2.2) in the case of the sequence  $h_\nu(\tau)$  (see (1.4)) the following

**Lemma 1.** *If*

$$(2.3) \quad |S(a, b)| \leq A(\Delta) \sqrt{a}, \quad \Delta \in (0, 1/6]$$

then  $(h_\nu = h_\nu(0))$

$$(2.4) \quad \begin{aligned} \sum_{T \leq h_{2\nu} \leq T+H} Z'[h_{2\nu}(\tau)] &= -\frac{1}{\pi} H \ln^2 P \cos \tau + \mathcal{O}(T^\Delta \ln^2 T), \\ \sum_{T \leq h_{2\nu+1} \leq T+H} Z'[h_{2\nu+1}(\tau)] &= \frac{1}{\pi} H \ln^2 P \cos \tau + \mathcal{O}(T^\Delta \ln^2 T), \end{aligned}$$

where  $\mathcal{O}$ -estimates are uniform for  $\tau \in [-\pi, \pi]$ .

*Proof.* We obtain from (2.2) by (1.4)

$$(2.5) \quad \begin{aligned} Z'[h_\nu(\tau)] &= 2(-1)^{\nu+1} \ln P \cos \tau - \\ &- 2 \sum_{2 \leq n \leq P} \frac{1}{\sqrt{n}} \ln \frac{P}{n} \cos\{\pi\nu - h_\nu(\tau) \ln n + \tau\} + \\ &+ \mathcal{O}(T^{-1/4} \ln T), \quad h_\nu(\tau) \in [T, T+H]. \end{aligned}$$

□

2.2. Since (see [5], (23))

$$(2.6) \quad \sum_{T \leq h_\nu \leq T+H} 1 = \frac{1}{2\pi} H \ln \frac{T}{2\pi} + \mathcal{O}(1) = \frac{1}{\pi} H \ln P + \mathcal{O}(1),$$

then we obtain from (2.5) (comp. [4], (59)-(61), [6], (51)-(53)) that

$$(2.7) \quad \sum_{T \leq h_\nu \leq T+H} Z'[h_\nu(\tau)] = -2\bar{w}(T, H; \tau) + \mathcal{O}(\ln^2 T),$$

where

$$\begin{aligned} \bar{w} = & \frac{1}{2}(-1)^{\bar{\nu}} \sum_n \frac{1}{\sqrt{n}} \ln \frac{P}{n} \cos \varphi + \\ & + \frac{1}{2}(-1)^{N+\bar{\nu}} \sum_n \frac{1}{\sqrt{n}} \ln \frac{P}{n} \cos(\omega N + \varphi) + \\ & + \frac{1}{2}(-1)^{\bar{\nu}} \sum_n \frac{1}{\sqrt{n}} \ln \frac{P}{n} \tan \frac{\omega}{2} \sin \varphi + \\ & + \frac{1}{2}(-1)^{N+\bar{\nu}+1} \sum_n \frac{1}{\sqrt{n}} \ln \frac{P}{n} \tan \frac{\omega}{2} \sin(\omega N + \varphi), \end{aligned}$$

where

$$\omega = \pi \frac{\ln n}{\ln P}, \quad \varphi = h_{\bar{\nu}}(\tau) \ln n - \tau, \quad n \in [2, P),$$

and

$$\bar{\nu} = \min\{\nu : h_{\nu} \in [T, T+H]\}, \quad \bar{\nu} + N = \max\{\nu : h_{\nu} \in [T, T+H]\}.$$

Of course, we have

$$\sum_{T \leq h_{\nu}(\tau) \leq T+H} 1 = \sum_{T \leq h_{\nu} \leq T+H} 1 + \mathcal{O}(1)$$

for any fixed  $\tau \in [-\pi, \pi]$ . Now, it is clear that the method [6], (54)-(64) implies by (2.3) that

$$\bar{w} = \mathcal{O}(T^{\Delta} \ln^2 T)$$

uniformly for  $\tau \in [-\pi, \pi]$ , and consequently we obtain (see (2.7)) the estimate

$$(2.8) \quad \sum_{T \leq h_{\nu} \leq T+H} Z'[h_{\nu}(\tau)] = \mathcal{O}(T^{\Delta} \ln^2 T)$$

uniformly for  $\tau \in [-\pi, \pi]$ .

2.3. Next, we have (see (2.5), (2.6))

$$\begin{aligned} \sum_{T \leq h_{\nu} \leq T+H} (-1)^{\nu} Z'[h_{\nu}(\tau)] &= -\frac{2}{\pi} H \ln^2 P \cos \tau - 2R + \mathcal{O}(\ln^2 P), \\ R &= \sum_{2 \leq n < P} \frac{1}{\sqrt{n}} \ln \frac{P}{n} \sum_{T \leq h_{\nu} \leq T+H} \cos\{h_{\nu}(\tau) \ln n - \tau\}. \end{aligned}$$

Since by (2.3) and [6], (65)-(79) we have the estimate

$$R = \mathcal{O}(T^{\Delta} \ln^2 T)$$

then we obtain the formula

$$(2.9) \quad \begin{aligned} \sum_{T \leq h_{\nu} \leq T+H} (-1)^{\nu} Z'[h_{\nu}(\tau)] &= \\ &= -\frac{2}{\pi} H \ln^2 P \cos \tau + \mathcal{O}(T^{\Delta} \ln^2 T) \end{aligned}$$

uniformly for  $\tau \in [-\pi, \pi]$ .

Finally, from (2.8), (2.9) formulae (2.4) follow.

### 3. INTEGRALS OVER DISCONNECTED SETS – LEMMA 2

Let (comp. [7], (3))

$$(3.1) \quad \begin{aligned} \mathbb{G}_{2\nu}(x) &= \{t : h_{2\nu}(-x) < t < h_{2\nu}(x), \quad t \in [T, T+H]\}, \quad x \in (0, \pi/2], \\ \mathbb{G}_{2\nu+1}(y) &= \{t : h_{2\nu+1}(-y) < t < h_{2\nu+1}(y), \quad t \in [T, T+H]\}, \quad y \in (0, \pi/2], \\ \mathbb{G}_1(x) &= \bigcup_{T \leq h_{2\nu} \leq T+H} \mathbb{G}_{2\nu}(x), \\ \mathbb{G}_2(y) &= \bigcup_{T \leq h_{2\nu+1} \leq T+H} \mathbb{G}_{2\nu+1}(y). \end{aligned}$$

The following lemma holds true.

**Lemma 2.** (2.3) implies

$$(3.2) \quad \begin{aligned} \int_{\mathbb{G}_1(x)} Z'(t) dt &= -\frac{2}{\pi} H \ln P \sin x + \mathcal{O}(xT^\Delta \ln T), \\ \int_{\mathbb{G}_2(y)} Z'(t) dt &= \frac{2}{\pi} H \ln P \sin y + \mathcal{O}(yT^\Delta \ln T). \end{aligned}$$

*Proof.* First of all we have (see (1.4), comp. [7], (51))

$$\left( \frac{dh_{2\nu}(\tau)}{d\tau} \right)^{-1} = \vartheta'_1[h_{2\nu}(\tau)] = \ln P + \mathcal{O}\left(\frac{H}{T}\right).$$

Next, from (2.2) by (2.3) we obtain the estimate

$$Z'(t) = \mathcal{O}(T^\Delta \ln^2 T), \quad t \in [T, T+H]$$

(Abel transformation). Then we have (comp. [7], (52)) that

$$(3.3) \quad \begin{aligned} \int_{-x}^x Z'[h_{2\nu}(\tau)] d\tau &= \int_{-x}^x Z'[h_{2\nu}(\tau)] \left( \frac{dh_{2\nu}(\tau)}{d\tau} \right)^{-1} \frac{dh_{2\nu}(\tau)}{d\tau} d\tau = \\ &= \ln P \int_{h_{2\nu}(-x)}^{h_{2\nu}(x)} Z'(t) dt + \mathcal{O}\left(x \frac{H}{T} T^\Delta \ln^2 T \frac{1}{\ln T}\right) = \\ &= \ln P \int_{\mathbb{G}_{2\nu}(x)} Z'(t) dt + \mathcal{O}(xHT^{-5/6} \ln T). \end{aligned}$$

Consequently, we obtain from the first formula in (2.4) by (2.6), (3.1), (3.3) the following asymptotic equality

$$\begin{aligned} \int_{\mathbb{G}_1(x)} Z'(t) dt &= -\frac{2}{\pi} H \ln P \sin x + \\ &+ \mathcal{O}(xT^\Delta \ln T) + \mathcal{O}(xH^2T^{-5/6} \ln^2 T), \end{aligned}$$

i.e. the first integral in (3.2). The second integral can be derived by a similar way.  $\square$

#### 4. AN ESTIMATE FROM BELOW – LEMMA 3

The following lemma holds true.

**Lemma 3.** From (2.3) the estimate

$$(4.1) \quad \int_T^{T+H} |Z'(t)| dt > \frac{4}{\pi} (1-\epsilon) H \ln P, \quad P = \sqrt{\frac{T}{2\pi}}, \quad H \in [T^{\Delta+\epsilon}, \sqrt[4]{T}]$$

follows, where  $\epsilon > 0$  is an arbitrarily small number.

*Proof.* Let (comp. [8], (10))

$$\begin{aligned} \mathbb{G}_1^+(x) &= \{t : Z'(t) > 0, \quad t \in \mathbb{G}_1(x)\}, \\ \mathbb{G}_1^-(x) &= \{t : Z'(t) < 0, \quad t \in \mathbb{G}_1(x)\}, \\ \mathbb{G}_1^0(x) &= \{t : Z'(t) = 0, \quad t \in \mathbb{G}_1(x)\}, \end{aligned}$$

and the symbols

$$\mathbb{G}_2^+(y), \mathbb{G}_2^-(y), \mathbb{G}_2^0(y)$$

have similar meaning. Of course

$$m\{\mathbb{G}_1^0(x)\} = m\{\mathbb{G}_2^0(y)\} = 0.$$

Since the expressions (3.2) in the case

$$H \in [T^{\Delta+\epsilon}, \sqrt[4]{T}], \quad x, y \in (0, \pi/2]$$

are asymptotic formulae then from them we obtain the following inequalities

$$\begin{aligned} (4.2) \quad & \frac{2}{\pi}(1-\epsilon)H \ln P < - \int_{\mathbb{G}_1(\pi/2)} Z'(t) dt \leq \\ & \leq - \int_{\mathbb{G}_1^-(\pi/2)} Z'(t) dt = \int_{\mathbb{G}_1^-(\pi/2)} |Z'(t)| dt, \\ & \frac{2}{\pi}(1-\epsilon)H \ln P < \int_{\mathbb{G}_2(\pi/2)} Z'(t) dt \leq \int_{\mathbb{G}_2^+(\pi/2)} |Z'(t)| dt. \end{aligned}$$

Since

$$\mathbb{G}_1^-(\pi/2) \cup \mathbb{G}_2^+(\pi/2) \subset [T, T+H], \quad \mathbb{G}_1^-(\pi/2) \cap \mathbb{G}_2^+(\pi/2) = \emptyset$$

then by (4.2) needful estimate

$$\begin{aligned} \int_T^{T+H} |Z'(t)| dt & \geq \int_{\mathbb{G}_1^-(\pi/2)} |Z'(t)| dt + \int_{\mathbb{G}_2^+(\pi/2)} |Z'(t)| dt > \\ & > \frac{4}{\pi}(1-\epsilon)H \ln P. \end{aligned}$$

follows. □

## 5. QUADRATURE FORMULA – LEMMA 4

The following lemma holds true.

**Lemma 4.** *On Riemann hypothesis we have the following asymptotic formula*

$$\begin{aligned} (5.1) \quad & \int_T^{T+H} |Z'(t)| dt = 2 \sum_{T \leq t_0 \leq T+H} |Z(t_0)| + \\ & + \mathcal{O}\left(T^{\frac{A}{\ln \ln T}}\right), \quad H \in [T^\mu, \sqrt[4]{T}], \end{aligned}$$

where  $0 < \mu$  is an arbitrary small number.

*Proof.* First of all, we have on Riemann hypothesis the following two Littlewood's estimates

$$(5.2) \quad \gamma'' - \gamma' < \frac{A}{\ln \ln \gamma'}, \quad \gamma' \rightarrow \infty$$

(see [2], p. 237), and

$$(5.3) \quad Z(t) = \mathcal{O}\left(t^{\frac{A}{\ln \ln t}}\right), \quad t \rightarrow \infty$$

(see [13], p. 300). Next, on Riemann hypothesis we have the following basic configuration (see Remark 2)

$$(5.4) \quad \gamma' < t_0 < \gamma''; \quad t_0 \in [T, T+H].$$

Now, there are following possibilities (see (5.4)): either

$$\begin{aligned} (5.5) \quad & Z(t) > 0, \quad t \in (\gamma', \gamma'') \Rightarrow \\ & Z'(t) > 0, \quad t \in (\gamma', t_0), \quad Z'(t) < 0, \quad t \in (t_0, \gamma''), \end{aligned}$$

or

$$(5.6) \quad \begin{aligned} Z(t) < 0, \quad t \in (\gamma', \gamma'') &\Rightarrow \\ Z'(t) < 0, \quad t \in (\gamma', t_0), \quad Z'(t) > 0, \quad t \in (t_0, \gamma''). \end{aligned}$$

Consequently, (5.5) and (5.6) imply that

$$(5.7) \quad \int_{\gamma'}^{\gamma''} |Z'(t)| dt = 2|Z(t_0)|, \quad \forall t_0 \in [T, T+H].$$

Similarly, we obtain (see (5.2), (5.3)) the estimates

$$(5.8) \quad \int_{\bar{\gamma}'}^{\bar{\gamma}''} |Z'(t)| dt, \quad \int_{\bar{\gamma}'}^{\bar{\gamma}''} |Z'(t)| dt = \mathcal{O}\left(\frac{T^{\frac{A}{\ln \ln T}}}{\ln \ln T}\right)$$

in the following cases

$$\bar{\gamma}' < T \leq t_0 < \bar{\gamma}'', \quad \bar{\gamma}' < t_0 \leq T+H < \bar{\gamma}''.$$

Now, our formula (5.1) follows from (5.7), (5.8).  $\square$

## 6. PROOF OF THEOREM

We use the following formula

$$(6.1) \quad \begin{aligned} \int_T^{T+H} \sqrt{1 + \{Z'(t)\}^2} dt &= \\ &= \int_T^{T+H} |Z'(t)| dt + \int_T^{T+H} \frac{1}{\sqrt{1 + \{Z'(t)\}^2} + |Z'(t)|} dt. \end{aligned}$$

Since

$$0 < \frac{1}{\sqrt{1 + \{Z'(t)\}^2} + |Z'(t)|} \leq 1$$

and

$$(6.2) \quad \left. \frac{1}{\sqrt{1 + \{Z'(t)\}^2} + |Z'(t)|} \right|_{t=t_0} = 1, \quad t_0 \in [T, T+H],$$

i.e. the inequality (6.2) holds true for the finite set of values, then the mean-value theorem gives

$$(6.3) \quad \int_T^{T+H} \frac{1}{\sqrt{1 + \{Z'(t)\}^2} + |Z'(t)|} dt = \Theta H, \quad \Theta = \Theta(T, H) \in (0, 1).$$

Next, we obtain by (4.1), (5.1), ( $\mu \leq \epsilon$ ), the inequality

$$(6.4) \quad \begin{aligned} \frac{4}{\pi}(1-\epsilon)H \ln P &< \int_T^{T+H} |Z'(t)| dt = \\ &= 2 \sum_{T \leq t_0 \leq T+H} |Z'(t_0)| + \mathcal{O}\left(T^{\frac{A}{\ln \ln T}}\right). \end{aligned}$$

Hence, by (6.1)-(6.4) the formula (1.5) follows for

$$(6.5) \quad H \in [T^{\Delta+\epsilon}, \sqrt[4]{T}].$$

Since the Riemann hypothesis implies Lindelöf hypothesis a it implies that  $\Delta = \epsilon$  (comp. [1], p. 89), then we obtain from (6.5) that

$$H = T^{2\epsilon}; \quad 2\epsilon \rightarrow \epsilon,$$

(see (1.5)).

#### APPENDIX A. INFLUENCE OF JACOB'S LADDERS

If

$$\varphi_1\{\widehat{[\dot{T}, T+H]}\} = [T, T+H],$$

then from (1.5) we obtain (see [10], (9.7)) the formula

$$(A.1) \quad \int_{\dot{T}}^{\widehat{T+H}} \sqrt{1 + \{Z'_{\varphi_1}[\varphi_1(t)]\}^2} \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt \sim \\ \sim \left\{ 2 \sum_{T \leq t_0 \leq T+H} |Z(t_0)| + \Theta H + \mathcal{O}\left(T^{\frac{A}{\ln \ln T}}\right) \right\} \ln T, \quad T \rightarrow \infty.$$

From (A.1) we obtain by mean-value theorem that

$$(A.2) \quad \int_{\dot{T}}^{\widehat{T+H}} \sqrt{1 + \{Z'_{\varphi_1}[\varphi_1(t)]\}^2} dt \sim \\ \sim \frac{\ln T}{\left| \zeta\left(\frac{1}{2} + i\alpha\right) \right|^2} \left\{ 2 \sum_{T \leq t_0 \leq T+H} \left| \zeta\left(\frac{1}{2} + it_0\right) \right| + \Theta H + \mathcal{O}\left(T^{\frac{A}{\ln \ln T}}\right) \right\}, \\ \alpha \in (\dot{T}, \widehat{T+H}).$$

*Remark 4.* Since we have (see [10], (8.5))

$$\rho\{[T, T+H]; [\dot{T}, \widehat{T+H}]\} \sim (1-c)\pi(T) > (1-\epsilon)(1-c)\frac{T}{\ln T}, \quad T \rightarrow \infty,$$

where  $\rho$  denotes the distance of corresponding segments and  $\pi(T)$  is the prime-counting function and  $c$  is the Euler constant, then the formula (A.2) gives strongly non-local expression for the integral on the left-hand side of (A.2).

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